

# Vortex Lattice Inhomogeneity in Spatially Inhomogeneous Superfluids

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A trapped degenerate Bose gas exhibits superfluidity with spatially nonuniform superfluid density. We show that the vortex distribution in such a highly inhomogeneous rotating superfluid is nevertheless nearly uniform. The inhomogeneity in vortex density, which diminishes in the rapid-rotation limit, is driven by the discrete way vortices impart angular momentum to the superfluid. This effect favors highest vortex density in regions where the superfluid density is most uniform (e.g., the center of a harmonically trapped gas). A striking consequence of this is that the boson velocity deviates from a rigid-body form exhibiting a radial-shear flow past the vortex lattice.

It has long been understood that a superfluid can only rotate by nucleating quantized vortices [1, 2], which in turn control most of its macroscopic properties. In recent years, rapid progress in the field of confined degenerate Bose gases has led to the experimental realization of large vortex lattices [3, 4, 5, 6, 7].

One of the most striking features of these arrays is their apparent *uniformity*, despite the strong spatial variation of the local superfluid density imposed by the trap. These observations cannot be simply explained by the strong vortex interaction, nor by appealing to the imposed rigid-body rotation. Based on purely energetic considerations, one would expect a highly nonuniform vortex distribution that is suppressed at the center of the trap where the superfluid density (and therefore kinetic energy cost), as well as vortex repulsion, are largest [8]. Despite some attempts to understand this uniformity [9], which has also been observed in simulations [10], no clear physical explanation has appeared in the literature.

In this paper we present a theory of vortices in a confined, spatially inhomogeneous rotating superfluid. We provide a simple physical explanation for, and compute corrections to, a uniform vortex array. Our main result is the vortex density  $\bar{n}_v(r)$  which, in the simplest case of a harmonic trap with a strong uniaxial anisotropy and within the Thomas-Fermi (TF) limit, is given by

$$\bar{n}_v \simeq \frac{\omega}{\pi} - \frac{1}{2\pi} \frac{R^2}{(R^2 - r^2)^2} \ln \frac{e^{-1}}{\xi^2 \omega}, \quad \text{for } r \ll R, \quad (1)$$

with  $R$  the TF radius,  $\omega = \Omega m / \hbar$  the rescaled rotational velocity  $\Omega$ ,  $m$  the boson mass, and  $\xi$  the coherence length [11].

The nearly uniform vortex distribution  $\bar{n}_v \approx \omega / \pi$  is a consequence of a balance between spatial variations of the kinetic energy per vortex and the vortex chemical potential, both of which scale with the local superfluid density,  $\rho_s(r)$ . While it is energetically costlier to position vortices in a region where  $\rho_s(r)$  is high (the center of the trap), the vortex chemical potential (controlled by  $\rho_s(r)\omega$ ) is also high there, compensating and leading to an approximately uniform vortex density.

A spatially-dependent correction to  $\bar{n}_v(r)$  in Eq. (1) arises from vortex discreteness and the related inability of the vortex state to locally reproduce uniform vorticity

corresponding to rigid-body rotation. In an inhomogeneous condensate, the associated kinetic energy-density cost is spatially dependent, and is lowered by a nonuniform vortex distribution. The reduction in  $n_v(r)$  scales with  $\nabla^2 \ln \rho_s$ , i.e., it is smallest where the condensate is most uniform, and leads to the strongest vortex density suppression away from the center of the trap. The correction vanishes in the uniform condensate ( $R \rightarrow \infty$ ) and, (within the London approximation) dense vortex (fast rotation,  $\omega \xi^2 \simeq 1$ ) limits, in which condensate inhomogeneity and vortex discreteness are (seemingly) unimportant [12].

An immediate interesting consequence of the radial vortex lattice distortion is that the corresponding azimuthal superfluid velocity  $\mathbf{v}_s(r)$  deviates from a rigid-body form, exhibiting radial-shear flow. We expect that this will induce an azimuthal shear distortion of the lattice, with chirality set by the sense of the imposed rotation [13]. Below we sketch the derivation of these results.

A rotating superfluid is most easily analyzed in the frame in which the boundary conditions (i.e. the proverbial bucket) are stationary. For experiments on trapped Bose gases, this is the frame rotating with frequency  $\Omega$  (in which the normal fluid is stationary [2]). For simplicity we focus on a trap with a high degree of uniaxial anisotropy, which reduces the problem to two-dimensions perpendicular to the ( $z$ -) axis of rotation. Deep in the superfluid state the London description, which focuses on the superfluid phase  $\theta$  degree of freedom, is sufficient and is represented by the energy

$$E = \frac{\hbar^2}{2m} \int d^2r \rho_s(r) [(\nabla\theta)^2 - 2\omega(\hat{z} \times \mathbf{r}) \cdot \nabla\theta], \quad (2)$$

where within the TF approximation  $\rho_s(r) \approx (\mu - V(r))/g$ , with  $\mu$  the boson chemical potential,  $g$  the s-wave scattering potential and  $V(r) = V_T(r) - \frac{1}{2}m\Omega^2 r^2$  a combination of the trapping ( $V_T(r)$ ) and “centrifugal” potentials.

Before addressing the many vortex problem, we present the solution for a single vortex in an inhomogeneous superfluid, which, despite a number of studies [14, 15, 16, 17, 18], has eluded a complete solution. For a vortex located at  $\mathbf{r}_0$  off the trap center, the superfluid velocity  $\mathbf{v}_s = (\hbar/m)\nabla\theta$  is determined by the Euler-Lagrange (EL) equation for Eq. (2) (expressing boson number con-

servation)  $\nabla \cdot (\rho_s(r) \nabla \theta) = 0$ , subject to the vorticity constraint  $\nabla \times \nabla \theta = 2\pi \delta^{(2)}(\mathbf{r} - \mathbf{r}_0)$ , and a boundary condition of a vanishing superflow transverse to the boundary. In contrast to uniform systems found in condensed matter experiments (e.g., helium in a bucket), in an atomic trap  $\rho_s(r)$  vanishes at the boundary, thereby automatically satisfying the boundary condition. The solution can then be expressed as  $\theta = \theta_v + \theta_a$  with  $\nabla \theta_v = \frac{\hat{z} \times (\mathbf{r} - \mathbf{r}_0)}{(\mathbf{r} - \mathbf{r}_0)^2}$  the usual (uniform-superfluid) vortex form ensuring the topological vorticity constraint, and the single-valued analytic phase  $\theta_a$  satisfying the EL equation:

$$\nabla \rho_s \cdot \nabla \theta_a + \rho_s \nabla^2 \theta_a = -\nabla \rho_s \cdot \nabla \theta_v. \quad (3)$$

We reserve the full analysis of Eq. (3) to Ref. 19, focusing here on main results. Near  $\mathbf{r}_0$ , we find, in agreement with Refs. 14, 16,

$$\nabla \theta_a(\mathbf{r}) \approx \frac{\hat{z} \times \nabla \rho_s(r_0)}{2\rho_s(r_0)} \ln |\mathbf{r} - \mathbf{r}_0|/R. \quad (4)$$

Far away from the vortex,  $\mathbf{v}_s^a \equiv (\hbar/m) \nabla \theta_a(\mathbf{r}) \approx 0$ , vanishing like a dipole field with negative and positive vortices at  $\mathbf{r}_0$  and at the center of the trap ( $\mathbf{r} = 0$ ), respectively. Because for a typical trap  $\nabla \rho_s(r_0) \propto -\hat{\mathbf{r}}_0$ , and the logarithm is negative for  $\mathbf{r} \rightarrow \mathbf{r}_0$ , the analytic distortion in the superfluid velocity is perpendicular to  $\mathbf{r}_0$  and leads to a superflow that is no longer purely azimuthal around an off-axis vortex. In agreement with local mass conservation,  $v_s$  is smaller on the trap-center side of the vortex (where  $\rho_s(r)$  is larger) and larger on its outside (where  $\rho_s(r)$  is smaller). A refinement of experiments [3, 20] that have demonstrated the ability to measure the phase variation  $\theta(\phi)$  around a vortex should allow a direct detection of the superflow distortions predicted here.

In an ideal superfluid and in the absence of other forces, a vortex moves with the local superfluid velocity. Therefore, the above result has interesting implications for vortex dynamics. Namely, since  $\mathbf{v}_s^a(r)$  is finite at the center of the vortex (where  $|\mathbf{r} - \mathbf{r}_0| \approx \xi$ ), we find a remarkable result: a single vortex at radius  $r$ , without (sustained) externally imposed rotation will precess about the trap center at a frequency  $\omega \approx (\hbar/2mr)(\partial_r \rho_s(r)/2\rho_s(r)) \ln \frac{\xi}{R} \approx (\hbar/2mR^2) \ln \frac{R}{\xi}$ , that away from the condensate edges is roughly independent of  $r$ . Such vortex precession has in fact been seen experimentally [21], with a quality factor of order 10. The implication of the superflow distortion  $\mathbf{v}_s^a(r)$  is even richer for dynamics of a pair of vortices. We predict that two same-charge vortices will orbit their center of charge, that will in turn precess about the center of the trap, with two frequencies determined by vortex separation and location relative to the axis of the trap [19].

We now turn to the many-vortex problem, with the goal of computing the vortex spatial distribution in an inhomogeneous rotating superfluid. The total  $\mathbf{v}_s$ , measured in the laboratory frame, due to an array of  $N$  vortices is the sum of the contributions from each vortex:

$$\mathbf{v}_s(\mathbf{r}) = \frac{\hbar}{m} \nabla \theta = \frac{\hbar}{m} \sum_{i=1}^N \frac{\hat{z} \times (\mathbf{r} - \mathbf{r}_i)}{(\mathbf{r} - \mathbf{r}_i)^2}, \quad (5)$$

with vortex positions  $\mathbf{r}_i$  static in the frame of the normal component. In the above, we have neglected  $\mathbf{v}_s^a(\mathbf{r})$ , as it has a subdominant effect in the many-vortex problem [19]. The corresponding vortex density is given by  $n_v(r) = (2\pi)^{-1} \nabla \times \nabla \theta = \sum_i^N \delta^2(\mathbf{r} - \mathbf{r}_i)$ .

For large  $\Omega$  the vortex state is dense, and we can neglect the discrete vortex nature [embodied by Eq. (5)] and approximate  $\mathbf{v}_s(\mathbf{r})$  and  $n_v(\mathbf{r})$  by arbitrary smooth functions. Expressing  $E$  in Eq. (2) in terms of  $\mathbf{v}_s(\mathbf{r})$  (and dropping a constant), we have

$$E \simeq \frac{m}{2} \int d^2r \rho_s(r) (\mathbf{v}_s - \Omega(\hat{z} \times \mathbf{r}))^2, \quad (6)$$

which is clearly minimized by the rigid-body solution  $\mathbf{v}_s = \Omega \hat{z} \times \mathbf{r}$  corresponding to a *uniform* vortex density  $\bar{n}_{v0} = \omega/\pi = m\Omega/\pi\hbar$ .

Away from this classical rapid-rotation limit, vortex discreteness begins to matter and the above solution clearly breaks down, as  $\mathbf{v}_s(\mathbf{r})$  diverges as  $1/|\mathbf{r} - \mathbf{r}_j|$  near each vortex at  $\mathbf{r}_j$ ; it thus strongly deviates from rigid-body flow. In this regime, where a superfluid exhibits its locally irrotational quantum nature, the summation in Eq. (5) can no longer be replaced by an integration, and the minimization of  $E$  must be done directly over the  $\mathbf{r}_i$ , rather than over a field  $\mathbf{v}_s(\mathbf{r})$ .

For a *uniform* infinite condensate the problem was solved long ago by Tkachenko [22], who found that the solution is a hexagonal lattice characterized by the vortex density  $\bar{n}_{v0}$ . Even in the uniform case, for a finite system vortex discreteness manifests itself in the lower-critical rotational velocity  $\Omega_{c1} \approx (\hbar/mR^2) \ln \frac{R}{\xi}$  below which no rotation is supported by the condensate.

To analyze an inhomogeneous condensate, it is essential to faithfully incorporate vortex *discreteness* in treating the sum in Eq. (5). For  $\mathbf{r}$  near a vortex located at  $\mathbf{r}_j$ , the flow is dominated by a diverging contribution from the  $j$ -th vortex, with other vortices giving a subleading correction that is smooth for  $\mathbf{r} \rightarrow \mathbf{r}_j$ . With this observation  $\mathbf{v}_s(\mathbf{r}_j + \delta\mathbf{r})$  near  $\mathbf{r}_j$  is well-approximated by

$$\mathbf{v}_s(\mathbf{r}_j + \delta\mathbf{r}) \simeq \frac{\hbar}{m} \frac{\hat{z} \times \delta\mathbf{r}}{\delta r^2} + \bar{\mathbf{v}}_s(\mathbf{r}_j), \quad (7)$$

with the smooth superflow  $\bar{\mathbf{v}}_s(\mathbf{r})$

$$\bar{\mathbf{v}}_s(\mathbf{r}_j) \simeq \frac{\hbar}{m} \int d^2r' \bar{n}_v(\mathbf{r}') \frac{\hat{z} \times (\mathbf{r}_j - \mathbf{r}')}{(\mathbf{r}_j - \mathbf{r}')^2}, \quad (8)$$

due to all other vortices, expressed through a coarse-grained vortex density  $\bar{n}_v(\mathbf{r}) = \bar{n}_{v0}(1 - \nabla \cdot \mathbf{u}(\mathbf{r}))$ , or equivalently vortex displacement  $\mathbf{u}(\mathbf{r})$  that are to be determined [19]. Interestingly, and not unlike the special treatment of the discrete ( $k = 0$ ) ground-state BEC mode,

to retain vortex discreteness we extracted the dominant contribution to the flow around the  $j$ -th vortex and then safely replaced the rest of the sum by an integral over the coordinates of the other vortices. By taking a curl with respect to  $\mathbf{r}_j$  and solving for  $\bar{\mathbf{v}}_s(\mathbf{r})$ , Eq. (8) gives

$$\bar{\mathbf{v}}_s(\mathbf{r}) = \Omega[\hat{z} \times \mathbf{r} - 2\hat{z} \times \mathbf{u}(\mathbf{r})], \quad (9)$$

describing the deviation of the mean-field flow from the rigid-body velocity due to the distortion  $\mathbf{u}(\mathbf{r})$  from a uniform vortex lattice.

To compute the optimum vortex density  $n_v(\mathbf{r})$ , we express the total energy  $E$  of a vortex array [i.e., Eq. (6)] as a sum over lattice cells [23, 24], with each associated with a single vortex. Using Eqs. (7,9) for  $\mathbf{v}_s$  inside a cell, making a circular-cell (of area  $1/\bar{n}_v(\mathbf{r}_j)$ ) approximation, and assuming  $\rho_s(r)$  does not vary appreciably on the scale of the vortex spacing, we compute the energy per cell. The remaining sum over cells can be easily done, innocuously approximating it by an integral, i.e.,  $\sum_i \rightarrow \int d^2r n_v(\mathbf{r}) \approx \int d^2r \bar{n}_v(\mathbf{r})$ . Thus we obtain

$$E \simeq \int d^2r \rho_s(r) \left[ \omega(1 - \nabla \cdot \mathbf{u}) \ln \frac{1}{\xi^2 \omega(1 - \nabla \cdot \mathbf{u})} + 4\omega^2 u^2 \right], \quad (10)$$

where we have discarded  $u$ -independent terms.

Minimizing  $E[\mathbf{u}(\mathbf{r})]$  over  $\mathbf{u}(\mathbf{r})$ , and re-expressing the solution in terms of  $\bar{n}_v(\mathbf{r})$ , we finally find

$$\bar{n}_v(\mathbf{r}) = \frac{\omega}{\pi} + \frac{1}{8\pi} \nabla \cdot \left( \frac{1}{\rho_s(r)} \nabla \left[ \rho_s(r) \ln \frac{e^{-1}}{\pi \xi^2 \bar{n}_v(\mathbf{r})} \right] \right), \quad (11)$$

that gives a local vortex density in a rotated trapped superfluid, characterized by a rotation rate  $\Omega$  and local superfluid density  $\rho_s(r)$ , in agreement with recent lowest Landau level results [12, 25, 26, 27]. For a smoothly-varying  $\rho_s(r)$ , we expect that vortices will have a *locally* hexagonal lattice structure [22], but with a lattice parameter  $a(r) = (2/\sqrt{3}\bar{n}_v(r))^{1/2}$  that varies with radius. Equation (11) consists of a uniform contribution corresponding to rigid-body rotation of the superfluid and a correction that depends crucially on the  $\rho_s(r)$  profile and therefore on the shape of the trapping potential. As is clear from its derivation, this spatial variation of the vortex density arises from an interplay of the nonuniform trapping potential and vortex discreteness, the latter a fundamentally quantum-mechanical effect.

For a rotating condensate of size  $R$ , trapped in a smooth concave potential,  $\rho_s(r)$  varies on the scale  $R$ , leading to a *negative* correction (i.e. the second term in Eq. (11)) to the rigid-body vortex distribution, that vanishes in the thermodynamic limit as  $1/R^2$ . Therefore, (as depicted in Fig.1) we predict, generically, a vortex distribution that is denser at the center of the trap and falls off towards the condensate edge as  $\nabla(\nabla \rho_s(r)/\rho_s(r))$ . Within our London approximation, this nonuniformity also vanishes in the fast rotation ( $\omega \xi^2 \simeq 1$ ), dense-vortex limit, in which vortex cores overlap and their discreteness is unimportant [12].

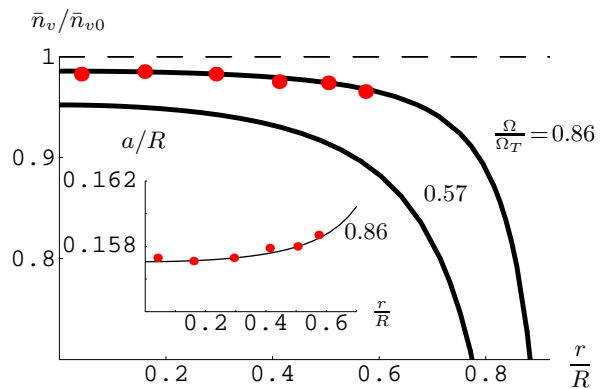


FIG. 1: (Color online) Main: Vortex density as a function of radius (normalized to  $\bar{n}_{v0}$ , dashed line) for these  $\Omega$  and  $R$  values (labelled by  $\Omega$ ):  $\Omega = .86\Omega_T$  and  $R = 49\mu\text{m}$ ;  $\Omega = .57\Omega_T$  and  $R = 31\mu\text{m}$ ; points are experimental values for the  $\Omega = .86\Omega_T$  case adapted from Ref. 28. Inset: Hexagonal lattice parameter  $a/R$  as a function of radius for the  $\Omega = .86\Omega_T$  case; points correspond to same data as in main.

Recalling that a *uniform* vortex distribution corresponds to a rigid-body rotational superflow,  $\bar{\mathbf{v}}_s(\mathbf{r}) = \Omega \hat{z} \times \mathbf{r}$ , the vortex lattice distortion predicted in Eq. (11) has an immediate remarkable consequence, namely a radial shear of the superfluid velocity, with the average rotational frequency decreasing with radius. As a result, unlike a uniform lattice, a radially distorted vortex lattice rotating at rate  $\Omega$  cannot be stationary with respect to the surrounding shearing superfluid. Symmetry arguments then suggest that for a nonideal superfluid, a vortex lattice should exhibit a chiral azimuthal radial-shear distortion in the direction opposite to the fluid flow [13].

We can use Eq. (11) to predict the radial vortex density for a rotating harmonically trapped BEC with trap frequency  $\Omega_T$ . Deep below  $T_c$  for a harmonic trap and away from the condensate edges we expect  $\rho_s(r)$  to be well approximated by the TF expression  $\rho_s(r) \approx \rho_s^0(1 - r^2/R^2)$ . Here,  $R^{-2} = m(\Omega_T^2 - \Omega^2)/2\mu$  is reduced by the applied rotation; however, we shall regard it as an experimentally given parameter. Together with a mild approximation  $\bar{n}_v(r) \approx \omega/\pi$  in the argument of the logarithm, we immediately obtain Eq. (1), which in Fig. 1 we plot for  $^{87}\text{Rb}$  using realistic [28] parameters of  $\Omega = .86\Omega_T$ ,  $R = 49\mu\text{m}$  (top curve), and  $\Omega = .57\Omega_T$ ,  $R = 31\mu\text{m}$  (bottom curve) along with data from Ref. 28 for the former case. The inset shows  $a$  for the  $\Omega = .86\Omega_T$  case. Here,  $\xi = \sqrt{\hbar/m\Omega_T}$  is the TF value [29] and  $\Omega_T = 52\text{s}^{-1}$ . This agreement with experimental data is achieved with *no* adjustable parameters.

A combination of magnetic and optical trapping allows an unparalleled degree of control over the single-body potential seen by the atoms. A wide range of experimentally accessible trapping potentials allows stringent tests of our predictions. Consider, for example, a sombrero potential,  $V(r)$ , consisting of a shallow overall trap with frequency  $\Omega_T$  and a repulsive Gaussian of width  $\ell$ , with  $\ell \ll R$ . The superfluid density will display a cor-

responding “dip” near the trap center, which within the TF approximation is given by

$$\rho_s(r) = \rho_{s0} - \rho_{s1} \exp(-r^2/2\ell^2), \quad (12)$$

with  $\rho_{s0}$  ( $> \rho_{s1}$ ) arising from the shallow part of the trap and therefore approximately constant on the scale  $\ell$ . Using this  $\rho_s(r)$  inside Eq. (11), we find a vortex density

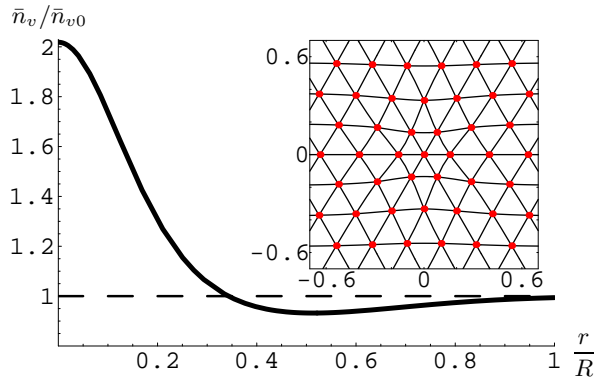


FIG. 2: (Color online) Main: The solid line depicts the vortex density (normalized to  $\bar{n}_{v0}$ ; dashed line) as a function of radius for the condensate profile Eq. (12). Inset: Locations of vortices (with lines to guide the eye).

profile  $n_v(r)$ , that is, interestingly, non-monotonic and near the center of the trap exceeds the asymptotic rigid-body value[19]. We plot  $\bar{n}_v(r)$  for a large, slowly rotating condensate ( $R = 500\mu\text{m}$ ,  $\Omega = .2\text{s}^{-1}$ ,  $\rho_{s1}/\rho_{s0} = 0.67$ , and

$\ell = .3R$ ) in Fig. 2. The growth in  $\bar{n}_v(r)$  near the center of the trap manifests itself as contracted inner vortex rings, as shown in the inset. Unfortunately, current experiments, which work with relatively small condensates and vortex lattices, quickly run out of length scales to quantitatively test details of a spatially varying vortex distribution. We hope, however, that this qualitatively distinctive vortex response to a tunable trap potential will be observable in the next generation experiments.

To conclude, we have studied the vortex spatial distribution in a rotating trapped superfluid and showed that, consistent with experiments and despite a strongly inhomogeneous condensate, in the limit of high rotation rates and a large condensate, vortex density is nevertheless nearly uniform. We have computed the leading spatially-dependent correction to  $\bar{n}_v(r)$ , which arises from an interplay of an inhomogeneous trap potential and vortex discreteness, and showed that *generically* it leads to a vortex density that is largest at the center of the trap, in striking contrast to simple energetic expectations. We hope that our predictions, which are quantitatively and qualitatively consistent with recent experiments[28], will stimulate more detailed experimental work on inhomogeneous vortex states.

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invalid near  $R$ . Although in a more careful approximation this divergence does not occur, we expect that our small gradient (in  $\rho_s(r)$ ) analysis breaks down near the edge of the condensate where the correction to the uniform, rigid-body rotation distribution is large.

- [12] Although the vortex lattice distortion appears to vanish in the rapid rotation ( $\omega\xi^2 \approx 1$ ) limit, this is an artifact of our London approximation that breaks down in this dense vortex limit. As was first shown by Ho [25], in this regime a complementary lowest Landau level (LLL) approximation applies and predicts a relation  $\bar{n}_v(r) = \omega/\pi + (4\pi)^{-1}\nabla^2 \ln \rho_s(r)$ , that (up to the logarithmic factor  $\propto \ln(\xi^2\omega)$ ) is in agreement with our prediction. In contrast to an assumption by Ho of a uniform  $\bar{n}_v$  (that leads to a Gaussian condensate profile  $\rho_s(r)$ ), subsequent to our work it was demonstrated [26, 27] that indeed the vortex lattice distortion extends to the LLL regime and leads to a Thomas-Fermi condensate profile in agreement with our predictions.
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